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Quantification of global properties of symmetric Markov processes

By

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Abstract

We summarize recent results on upper/lower rate functions for symmetric Markov processes generated by regular Dirichlet forms. These functions are quantitative expressions of conservativeness/transience. We give upper/lower rate functions in terms of the growth rates of the volume and coefficient.

§ 1. Introduction

We are concerned with the global sample path properties of symmetric Markov processes generated by regular Dirichlet forms. In particular, we would like to find the quantitative characterizations of these properties. Such characterizations are expressed in terms of upper/lower rate functions, which are well known for Brownian motions, symmetric stable processes and symmetric diffusion processes as will be mentioned below. The present article is a summary of the results, which are obtained in [39, 40], on upper/lower rate functions for symmetric Markov processes.

We first explain the notions of upper/lower rate functions by using the Brownian motion on $\mathbb{R}^d$. Let $(\{B_t\}_{t\geq 0}, P)$ be the Brownian motion on $\mathbb{R}^d$ starting from the origin. Here P and B_t denote, respectively, the law and the position at time t of the Brownian particle. Since

$$P(B_t \in E) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_E \exp\left(-\frac{|y|^2}{2t}\right) \mathrm{d}y$$

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for any Borel subset $E \subset \mathbb{R}^d$, we have $P\left(B_t \in \mathbb{R}^d\right) = 1$ for any $t > 0$. This implies that the Brownian motion is *conservative* in the sense that

$$P\left(|B_t| < \infty \text{ for any } t > 0\right) = 1.$$

We can further see the large time behaviors of sample paths depending on the space dimension d: if $d = 1, 2$, then the Brownian motion on $\mathbb{R}^d$ is *recurrent* in the sense that

$$P\left(\liminf_{t \to \infty} |B_t| = 0\right) = 1.$$

This means that the Brownian particle returns to any neighborhood of the origin infinitely often. On the other hand, if $d \geq 3$, then the Brownian motion on $\mathbb{R}^d$ is *transient* in the sense that

$$P\left(\lim_{t \to \infty} |B_t| = \infty\right) = 1,$$

that is, the Brownian particle escapes to infinity eventually.

Here we would like to understand conservativeness and transience in more detail. In order to do so, we consider the quantitative characterizations of these properties. As for conservativeness, we would like to find the upper bound of $|B_t|$ for all sufficiently large $t > 0$. *Kolmogorov's test* (e.g., see [27, 4.12]) gives us a dichotomy for such bound: if $g(t)$ is a positive increasing function on $(0, \infty)$ such that $g(t) \to \infty$ as $t \to \infty$, then

$$P\left(\text{there exists } T > 0 \text{ such that } |B_t| \leq \sqrt{t}g(t) \text{ for all } t \geq T\right) = 1 \text{ or } 0 \tag{1.1}$$

according as

$$\int_{\cdot}^{\infty} g(t)^d \exp\left(-\frac{g(t)^2}{2}\right) \frac{\mathrm{d}t}{t} < \infty \text{ or } = \infty.$$

When the probability in (1.1) is one, the function $R(t) = \sqrt{t}g(t)$ is called an *upper rate function* for the Brownian motion on $\mathbb{R}^d$. For example, the function $R(t) = \sqrt{(2+\varepsilon)t \log\log t}$ $(\varepsilon > -2)$ is an upper rate function for the Brownian motion on $\mathbb{R}^d$ if and only if $\varepsilon > 0$. As a consequence of this fact, we can get *Khintchine's law of the iterated logarithm*:

$$\limsup_{t \to \infty} \frac{|B_t|}{\sqrt{2t \log\log t}} = 1, \quad P\text{-a.s.}$$

As for transience, we would like to find the lower bound of $|B_t|$ for all sufficiently large $t > 0$. *Dvoretzky-Erdös' test* (e.g., see [9]) shows that for the transient case, if $g(t)$ is a positive decreasing function on $(0, \infty)$ such that $g(t) \to 0$ as $t \to \infty$, then

$$P\left(\text{there exists } T > 0 \text{ such that } |B_t| \geq \sqrt{t}g(t) \text{ for all } t \geq T\right) = 1 \text{ or } 0 \tag{1.2}$$

according as

$$\int_{\cdot}^{\infty} g(t)^{d-2} \frac{\mathrm{d}t}{t} < \infty \text{ or } = \infty.$$

When the probability in (1.2) is one, the function $r(t) = \sqrt{t}g(t)$ is called a *lower rate function* for the Brownian motion on $\mathbb{R}^d$. We can see that the function $r(t) = \sqrt{t}/(\log t)^{\frac{1+\varepsilon}{d-2}}$ ($\varepsilon > -1$) is a lower rate function for the Brownian motion on $\mathbb{R}^d$ if and only if $\varepsilon > 0$. Here we remark that the notion of lower rate functions is meaningful even for $d = 2$, and Spitzer [44] obtained the integral test on these functions (see also [27]).

The integral tests as mentioned above are further extended to (symmetric) stable processes ([30, 47, 48, 20, 31]). For the Brownian motions and stable processes on $\mathbb{R}^d$, the (time-space) scaling property plays an important role for the proof. Such integral tests are also proved for a class of symmetric diffusion processes ([25, 4]) and symmetric jump processes ([42]). These processes do not have the scaling property in general, but such difficulty was overcome by using the heat kernel estimates. We note that the integral tests in [42] are applicable to jump processes on fractals.

The purpose of this article is to discuss upper/lower rate functions for symmetric Markov processes generated by *Dirichlet forms*. Dirichlet form is an axiomatization of the Dirichlet integral and defined as closed Markovian symmetric forms on L^2 spaces (see, e.g., [5] and [12]). The Dirichlet form theory plays an important role in order to construct and analyze symmetric Markov processes. In particular, since we can construct symmetric Markov processes from Dirichlet forms under mild conditions, the Dirichlet form theory fits several singularities related to the generators and the underlying spaces. However, it is a highly non-trivial problem in general to deduce the sample path properties of symmetric Markov processes from the analytic information of associated Dirichlet forms. Our discussion in this article is an attempt to attack this problem.

There are a number of results on the conservativeness and rate functions for symmetric diffusion processes generated by regular Dirichlet forms (see, e.g., [8, 13, 14, 26, 43, 45, 46] for conservativeness, [16, 17, 19, 21] for rate functions and [15] for survey). These results are applicable to more general symmetric diffusion processes than those in [4, 25], but we do not know the sharpness in general. Recently, conservativeness criteria are established for symmetric Markov processes with jumps by developing the approach for symmetric diffusion processes (see [18, 32, 33, 38, 41]). Especially for Markov chains on weighted graphs, we can find a conservativeness criterion and upper rate functions similar to those for the diffusion case (see [10, 23] for conservativeness, [22, 24] for upper rate functions, and [29] for survey). Our attempt is regarded as a continuation of these works.

Throughout this paper, the letters c and C (with subscript) denote finite positive constants which may vary from place to place. For nonnegative functions $f(x)$ and $g(x)$ on a space S, we write $f(x) \asymp g(x)$ if there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 g(x) \le f(x) \le c_2 g(x) \quad \text{for any } x \in S.$$

§ 2. Preliminaries

In this section, we recall the notions of Dirichlet forms and symmetric Markov processes by following [5] and [12].

§ 2.1. Analytic notions

We first recall the definition of Dirichlet forms and the expression formula. Let (X, d) be a locally compact separable metric space and m a positive Radon measure on X with full support. We denote by $C(X)$ the totality of continuous functions on X, and by $C_0(X)$ the totality of continuous functions on X with compact support. Let $\mathcal{F}$ be a dense linear subspace of $L^2(X;m)$ and $\mathcal{E}$ a symmetric bilinear form on $\mathcal{F} \times \mathcal{F}$. Then the pair $(\mathcal{E}, \mathcal{F})$ is called a *Dirichlet form* on $L^2(X;m)$ if the next three conditions hold:

(i) $\mathcal{E}(u,u) \geq 0$ for any $u \in L^2(X;m)$;

(ii) $\mathcal{F}$ is a real Hilbert space with the inner product $\mathcal{E}_1(\cdot,\cdot)$, where

$$(u,v)_{L^2} = \int_X u(x)v(x)\, m(\mathrm{d}x)$$

and

$$\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2} \quad \text{for } u,v \in \mathcal{F};$$

(iii) For any $u \in \mathcal{F}$, $v = 0 \vee u \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(v,v) \leq \mathcal{E}(u,u)$.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X;m)$. Then a nonpositive definite self-adjoint operator A on $L^2(X;m)$ is uniquely determined by the following relation:

$$\mathcal{F} = \mathcal{D}(\sqrt{-A}), \quad \mathcal{E}(u,v) = (\sqrt{-A}u, \sqrt{-A}v)_{L^2}, \quad u,v \in \mathcal{F}$$

([12, Theorem 1.3.1]). Moreover, the semigroup $T_t = e^{tA}$ generated by A becomes the *Markovian semigroup* ([12, Theorem 1.4.1]). Namely, for any $f \in L^2(X;m)$ such that $0 \leq f \leq 1$, m-a.e., we have $0 \leq T_t f \leq 1$, m-a.e.

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(X)$ is dense both in $\mathcal{F}$ with respect to the norm $\|u\|_{\mathcal{E}_1} = \sqrt{\mathcal{E}_1(u,u)}$, and in $C_0(X)$ with respect to the uniform norm $\|\cdot\|_\infty$. The *Beurling-Deny formula* gives us an expression of regular Dirichlet forms $(\mathcal{E}, \mathcal{F})$ ([12, Theorem 3.2.1, Lemma 4.5.4]): for any $u,v \in \mathcal{F} \cap C_0(X)$,

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \iint_{X \times X \setminus \mathrm{diag}} (u(x)-u(y))(v(x)-v(y))\, J(\mathrm{d}x\mathrm{d}y) + \int_X u(x)v(x)\, k(\mathrm{d}x).$$

Here $\mathrm{diag} = \{(x,y) \in X \times X \mid x = y\}$ and

- $(\mathcal{E}^{(c)}, \mathcal{F} \cap C_0(X))$ is a *strongly local* symmetric form. Namely, if the function v is constant on a neighborhood of $\mathrm{supp}[u]$, then $\mathcal{E}^{(c)}(u, v) = 0$;
- J is a symmetric and positive Radon measure on $X \times X \setminus \mathrm{diag}$;
- k is a positive Radon measure on X.

In particular, these three elements are uniquely determined for each regular Dirichlet form. The measures J and k are called, respectively, the *jumping measure* and *killing measure* associated with $(\mathcal{E}, \mathcal{F})$.

For any $u \in \mathcal{F} \cap C_0(X)$, there exists a unique positive Radon measure $\mu^c_{\langle u \rangle}$ on X such that

$$\mathcal{E}^{(c)}(u, u) = \frac{1}{2} \int_X \mathrm{d}\mu^c_{\langle u \rangle}$$

([12, p.126]). We call $\mu^c_{\langle u \rangle}$ the *local part of the energy measure* of u. Hence the Beurling-Deny formula gives us an integral representation of regular Dirichlet forms.

We say that a function u on X belongs *locally* to $\mathcal{F}$ ($u \in \mathcal{F}_{\mathrm{loc}}$ in notation) if for any relatively compact open set $G \subset X$, there exists $u_G \in \mathcal{F}$ such that $u = u_G$, m-a.e. on G. We can then extend the measure $\mu^c_{\langle u \rangle}$ to any $u \in \mathcal{F}_{\mathrm{loc}}$ ([12, p.130]).

Example 2.1. Assume that $X = \mathbb{R}^d$ and m is the d-dimensional Lebesgue measure $\mathrm{d}x$. Let $H^1(\mathbb{R}^d)$ be the Sobolev space of order 1 given by

$$H^1(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \mid \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^d), 1 \le i \le d \right\}.$$

Here we take the derivative in the sense of the Schwartz distribution. If we let $\mathcal{E}$ be the half of the Dirichlet integral $\mathbf{D}$, that is,

$$\mathcal{E}(u, v) = \frac{1}{2}\mathbf{D}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x,$$

then $(\mathcal{E}, H^1(\mathbb{R}^d))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ (see, e.g., [12, Example 1.4.1]) and the operator A is the half of the Laplacian:

$$A = \frac{1}{2}\Delta.$$

Let us consider the more general case than that mentioned above. Let $\{a_{ij}\}_{i,j=1}^d$ be a family of Borel measurable functions on $\mathbb{R}^d$ such that

(i) $a_{ij} = a_{ji}$ for any $i, j \in \{1, 2, \ldots, d\}$;

(ii) for some positive constants c_1 and c_2 with $c_1 \le c_2$,

$$c_1 |\xi|^2 \le \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \le c_2 |\xi|^2, \quad x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d \setminus \{0\}.$$

Let $\mathcal{E}$ be a symmetric bilinear form on $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ defined by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)\,\mathrm{d}x, \quad u,v \in H^1(\mathbb{R}^d).$$

Since $\mathcal{E}(u,u) \asymp \mathbf{D}(u,u)$, $(\mathcal{E}, H^1(\mathbb{R}^d))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ such that the jumping and killing measures vanish and

$$\mu^c_{\langle u\rangle}(\mathrm{d}x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\,\mathrm{d}x.$$

In particular, we can formally write the operator A corresponding to $(\mathcal{E}, H^1(\mathbb{R}^d))$ as

$$Au = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right).$$

Example 2.2. Assume that $X = \mathbb{R}^d$ and m is the d-dimensional Lebesgue measure $\mathrm{d}x$. Let $c(x,y)$ be a positive symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$c_1 \le c(x,y) \le c_2, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$$

for some positive constants c_1 and c_2 with $c_1 \le c_2$. We now fix $\alpha \in (0,2)$ and define

$$\mathcal{F} = \left\{ u \in L^2(\mathbb{R}^d) \mid \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y < \infty \right\}$$

and

$$\mathcal{E}(u,v) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}} (u(x)-u(y))(v(x)-v(y))\frac{c(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y, \quad u,v \in \mathcal{F}.$$

Then $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ such that $\mu^c_{\langle u\rangle} = 0$, $k = 0$ and

$$J(\mathrm{d}x\mathrm{d}y) = \frac{c(x,y)}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y.$$

In particular, if $c(x,y)$ is a constant $c_{d,\alpha}$ given by

$$c_{d,\alpha} = \frac{\alpha 2^{\alpha-2}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)},$$

then the operator A is given by the half of the fractional Laplacian of order α:

$$Au = -\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}}u$$

(see, e.g., [12, Example 1.4.1]).

We next recall the notions of conservativeness, recurrence and transience by following [12, Subsection 1.6]. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(X;m)$. As mentioned in [12, p.56], we can extend the Markovian (L^2-)semigroup $\{T_t\}_{t\geq 0}$ associated with $(\mathcal{E},\mathcal{F})$ to $L^\infty(X;m)$. We will use the same notation for the extended (L^∞-)semigroup. We say that $(\mathcal{E},\mathcal{F})$ is *conservative* if $T_t 1 = 1$, m-a.e. for any $t > 0$.

For any $t > 0$ and $f \in L^2(X;m)$, the integral

$$S_t f = \int_0^t T_s f \, \mathrm{d}s$$

can be well defined as the limit of the Riemann sum in the L^2-strong convergence. Then S_t becomes a bounded symmetric operator on $L^2(X;m)$. Furthermore, we can extend uniquely the operators S_t and T_t on $L^1(X;m) \cap L^2(X;m)$, respectively, to the bounded linear operators on $L^1(X;m)$. We will use the same notations for these extended (L^1-)semigroups.

Let $L^1_+(X;m) = \{u \in L^1(X;m) \mid u \geq 0,\ m\text{-a.e.}\}$ and

$$Gf = \lim_{N\to\infty} S_N f, \quad f \in L^1_+(X;m).$$

Definition 2.3. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(X;m)$.

(i) $(\mathcal{E},\mathcal{F})$ is *recurrent* if $Gf = 0$ or ∞, m-a.e. for any $f \in L^1_+(X;m)$.

(ii) $(\mathcal{E},\mathcal{F})$ is *transient* if $Gf < \infty$, m-a.e. for any $f \in L^1_+(X;m)$.

We see from [12, Lemma 1.5.1] that $(\mathcal{E},\mathcal{F})$ is transient if and only if there exists $f \in L^1_+(X;m)$ satisfying $m(\{x \in X \mid f(x) = 0\}) = 0$ such that $Gf < \infty$, m-a.e.

An m-measurable set $A \subset X$ is said to be (T_t)-*invariant* if $T_t(f\mathbf{1}_A) = \mathbf{1}_A T_t f$, m-a.e. for any $f \in L^2(X;m)$ and $t > 0$. $(\mathcal{E},\mathcal{F})$ is called *irreducible* if $m(A) = 0$ either $m(X \setminus A) = 0$ for any invariant set $A \subset X$. It is known by [12, Lemma 1.6.4] that any irreducible Dirichlet form is recurrent or transient.

Let $\mathcal{F}_e$ be the totality of m-measurable functions u on X such that $|u| < \infty$ m-a.e. and there exists a sequence $\{u_n\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$, m-a.e. on X and

$$\lim_{m,n\to\infty} \mathcal{E}(u_n - u_m, u_n - u_m) = 0.$$

This sequence is called an *approximating sequence* of u. For any $u \in \mathcal{F}_e$ and its approximating sequence $\{u_n\}$, the limit

$$\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$$

exists and does not depend on the choice of $\{u_n\}$ ([12, Theorem 1.5.2]). We call $(\mathcal{F}_e,\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E},\mathcal{F})$ ([12, p.41]). In particular, if $(\mathcal{E},\mathcal{F})$ is transient, then $\mathcal{F}_e$ is complete with respect to $\mathcal{E}$ ([12, Lemma 1.5.5]).

We finally introduce the notion of capacity by following [12, Chapter 2]. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(X;m)$. Denote by $\mathcal{O}$ the family of all open subsets of X. For $A \in \mathcal{O}$, we let

$$\mathcal{L}_A = \{u \in \mathcal{F} \mid u \geq 1 \quad m\text{-a.e. on } A\} \tag{2.1}$$

and

$$\mathrm{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u,u), & \mathcal{L}_A \neq \emptyset \\ \infty, & \mathcal{L}_A = \emptyset. \end{cases} \tag{2.2}$$

We then define the (1-order) *capacity* of a set $A \subset X$ by

$$\mathrm{Cap}(A) = \inf_{B \in \mathcal{O},\, A \subset B} \mathrm{Cap}(B).$$

For $A \subset X$, a statement depending on $x \in A$ is said to hold *q.e.* on A (abbreviation for "quasi everywhere") if there exists a set $N \subset A$ with $\mathrm{Cap}(N) = 0$ such that the statement holds for each $x \in A \setminus N$.

Suppose now that $(\mathcal{E},\mathcal{F})$ is transient. We then define the (0-order) *capacity* $\mathrm{Cap}_{(0)}(A)$ by replacing $\mathcal{F}$ and $\mathcal{E}_1$ in (2.1) and (2.2), respectively, with $\mathcal{F}_e$ and $\mathcal{E}$. It follows from [12, p.74] that if $\mathcal{L}_B \neq \emptyset$, then there exists a unique element $e_B^{(0)} \in \mathcal{L}_B$ such that

$$\mathrm{Cap}_{(0)}(B) = \mathcal{E}(e_B^{(0)}, e_B^{(0)}).$$

The function $e_B^{(0)}$ is called the *equilibrium potential* of B. We also see from [12, Theorem 2.1.6] that $\mathrm{Cap}_{(0)}(A) = 0$ if and only if $\mathrm{Cap}(A) = 0$ for any $A \subset X$.

We say that a positive Radon measure μ on X is of (0-*order*) *finite energy integral* ($\mu \in S_0^{(0)}$ in notation) if there exists $C > 0$ such that

$$\int_X |f| \, \mathrm{d}\mu \leq C\sqrt{\mathcal{E}(f,f)} \quad \text{for any } f \in \mathcal{F} \cap C_0(X).$$

Then any measure $\mu \in S_0^{(0)}$ charges no set of zero capacity and associates a unique element $U\mu \in \mathcal{F}_e$, which is called the (0-*order*) *potential* of μ, such that

$$\mathcal{E}(U\mu, v) = \int_X \tilde{v} \, \mathrm{d}\mu \quad \text{for any } v \in \mathcal{F}_e$$

([12, p.85]). For any compact set K, there exists a unique measure $\nu_K \in S_0^{(0)}$ with $\mathrm{supp}[\nu_K] \subset K$ such that $e_K^{(0)} = U\nu_K$ and

$$\mathrm{Cap}_{(0)}(K) = \mathcal{E}(e_K^{(0)}, e_K^{(0)}) = \nu_K(K) \tag{2.3}$$

(0-order version of [12, Lemma 2.2.6]). The measure ν_K is called the (0-*order*) *equilibrium measure* of K.

§ 2.2. Probabilistic notions

We first introduce the notions of Markov processes and Hunt processes by following [12, A.2]. Let (X, d) be a locally compact separable metric space and $X_\Delta = X \cup \{\Delta\}$ the one point compactification of X. We denote by $\mathcal{B}(X)$ the family of all Borel measurable subsets of X and $\mathcal{B}(X_\Delta) = \mathcal{B}(X) \cup \{B \cup \{\Delta\} \mid B \in \mathcal{B}(X)\}$.

We say that $\mathbf{M} = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{P_x\}_{x\in X_\Delta})$ is a (*normal*) *Markov process* on X if it satisfies the following conditions:

(M.1) (i) P_x is a probability measure on a measurable space $(\Omega, \mathcal{M})$ for each $x \in X_\Delta$;

(ii) $\{\mathcal{M}_t\}_{t\geq 0}$ is a family of sub-σ-fields of $\mathcal{M}$ such that $\mathcal{M}_t$ is increasing in t;

(iii) X_t is a measurable map from $(\Omega, \mathcal{M}_t)$ to $(X_\Delta, \mathcal{B}(X_\Delta))$ for each $t \geq 0$.

(M.2) For each $t \geq 0$ and $E \in \mathcal{B}(X)$, $P_x(X_t \in E)$ is a measurable map with respect to $x \in X$.

(M.3) For any $t, s \geq 0$, $E \in \mathcal{B}(X)$ and $x \in X$,

$$P_x(X_{t+s} \in E \mid \mathcal{M}_t) = P_{X_t}(X_s \in E), \quad P_x\text{-a.s.}$$

(M.4) For any $t \geq 0$, $P_\Delta(X_t = \Delta) = 1$.

(M.5) $P_x(X_0 = x) = 1$ for any $x \in X$.

Under the measure P_x, a particle starts from $x \in X$ at time $t = 0$ and moves according to the law P_x. We regard X_t as the position of the particle at time t, and $\mathcal{M}_t$ as the information about the trajectory of the particle over the time interval $[0, t]$. The condition **(M.3)** is the so-called *Markov property*. This means that if we have the information about the current position X_t, then the past and future are independent to each other.

We now impose the following conditions on the Markov process $\mathbf{M}$ on X:

(M.6) (i) $X_\infty(\omega) = \Delta$ for any $\omega \in \Omega$;

(ii) $X_t(\omega) = \Delta$ for all $t \geq \zeta(\omega)(= \inf\{t > 0 \mid X_t(\omega) = \Delta\})$;

(iii) For each $t \geq 0$, there exists a map $\theta_t : \Omega \to \Omega$ such that $X_{t+s}(\omega) = X_t(\theta_s\omega)$ for any $\omega \in \Omega$ and $s \geq 0$;

(iv) For each $\omega \in \Omega$, the map $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$ with the left limit in $(0, \infty)$.

We call ζ and θ_t, respectively, the *life time* and the *translation operator* of $\mathbf{M}$. We regard Δ as the “cemetery point” of the particle.

The $\{\mathcal{M}_t\}$-*stopping time* σ is a $[0,\infty]$-valued function on Ω such that $\{\sigma \le t\} \in \mathcal{M}_t$ for any $t \ge 0$ and $\mathcal{M}_\sigma$ is a sub-σ-field of $\mathcal{M}$ defined by

$$\mathcal{M}_\sigma = \{\Lambda \in \mathcal{M} \mid \Lambda \cap \{\sigma \le t\} \in \mathcal{M}_t \text{ for any } t \ge 0\}\,.$$

Let $\mathcal{P}(X_\Delta)$ be the totality of probability measures on X_Δ and P_μ the probability measure on $(\Omega, \mathcal{M})$ defined by $P_\mu(\Lambda) = \int_{X_\Delta} P_x(\Lambda)\,\mu(\mathrm{d}x)$ for $\Lambda \in \mathcal{M}$.

Definition 2.4. Let $\mathbf{M} = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}_{t\ge 0}, \{X_t\}_{t\ge 0}, \{P_x\}_{x\in X_\Delta})$ be a Markov process on X satisfying **(M.6)**.

(i) $\mathbf{M}$ is *strong Markov* if $\{\mathcal{M}_t\}_{t\ge 0}$ is right continuous (i.e., $\mathcal{M}_t = \cap_{s>t}\mathcal{M}_s$ for any $t \ge 0$) and for any stopping time σ,

$$P_\mu(X_{\sigma+s} \in E \mid \mathcal{M}_\sigma) = P_{X_\sigma}(X_s \in E)$$

for any $\mu \in \mathcal{P}(X_\Delta)$, $E \in \mathcal{B}(X_\Delta)$ and $s \ge 0$.

(ii) $\mathbf{M}$ is *quasi-left-continuous* if for any sequence of stopping times $\{\sigma_n\}$ increasing to σ,

$$P_\mu\left(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma, \sigma < \infty\right) = P_\mu(\sigma < \infty), \quad P_\mu\text{-a.s.}$$

for any $\mu \in \mathcal{P}(X_\Delta)$.

(iii) $\mathbf{M}$ is a *Hunt process* on X if $\mathbf{M}$ is strong Markov and quasi-left-continuous.

We next explain relations between symmetric Hunt processes and Dirichlet forms. In what follows, we suppose that $\mathbf{M} = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}_{t\ge 0}, \{X_t\}_{t\ge 0}, \{P_x\}_{x\in X_\Delta})$ is a Hunt process on X. For $f \in \mathcal{B}(X)$, we define

$$p_t f(x) = E_x[f(X_t)]\left(= \int_\Omega f(X_t(\omega))\,P_x(\mathrm{d}\omega)\right) \quad \text{for } t \ge 0 \text{ and } x \in X$$

if the right hand side(=the expectation of $f(X_t)$ with respect to P_x) makes sense. Here we make the convention that $f(\Delta) = 0$. Then by the Markov property **(M.3)**, we obtain

$$p_t(p_s f) = p_{t+s} f \quad \text{for } t, s > 0$$

for any $f \in \mathcal{B}_b(X)$. We also see that

$$p_t f(x) = \int_X p_t(x, \mathrm{d}y) f(y)$$

for some Markovian kernel $p_t(x, \mathrm{d}y)$ (*Markovian* means that $p_t(x, E) \le 1$ for any $E \in \mathcal{B}(X)$). Using this kernel, we can define the *transition function of the Hunt process* $\mathbf{M}$ by

$$p_t(x, E) = P_x(X_t \in E) \quad \text{for } t \ge 0,\ x \in X \text{ and } E \in \mathcal{B}(X).$$

A Hunt process $\mathbf{M}$ is said to be *m-symmetric* if $\{p_t\}_{t>0}$ satisfies

$$\int_X p_t f(x)g(x)\,m(\mathrm{d}x) = \int_X f(x)p_t g(x)\,m(\mathrm{d}x)$$

for any $t > 0$ and $f, g \in \mathcal{B}_b(X)$. We can then extend $\{p_t\}_{t>0}$ to an L^2-semigroup by using the fact that $\int_X (p_t f)^2\,\mathrm{d}m \le \int_X f^2\,\mathrm{d}m$ for any $f \in L^2(X;m) \cap \mathcal{B}_b(X)$ (see, e.g., [12, p.30]). By using the same notation for the extended semigroup, we can define a (not necessarily regular) Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X;m)$ by

$$\mathcal{F} = \left\{ u \in L^2(X;m) \mid \lim_{t\to 0} \frac{1}{t}\int_X (u - p_t u)u\,\mathrm{d}m < \infty \right\},$$
$$\mathcal{E}(u,u) = \lim_{t\to 0} \frac{1}{t}\int_X (u - p_t u)u\,\mathrm{d}m.$$

On the other hand, it follows from so-called *Fukushima's theorem* ([11], [12, Chapter 7]) that for a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X;m)$, there exists an m-symmetric Hunt process $\mathbf{M}$ on X such that

$$T_t f(x) = p_t f(x), \quad m\text{-a.e.}, \quad f \in L^2(X;m) \cap \mathcal{B}_b(X).$$

In particular, the jumping measure and killing measure of $(\mathcal{E}, \mathcal{F})$ express the intensities of the jumps and killing inside, respectively ([12, 4.5]). Here the killing inside is the event that $\{\zeta < \infty,\ \lim_{t\to\zeta-0} X_t \in X\}$.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X;m)$ and $\mathbf{M}$ an associated m-symmetric Hunt process on X. Then by [12, Exercise 4.5.1], $(\mathcal{E}, \mathcal{F})$ is conservative if and only if so is $\mathbf{M}$, that is,

$$P_x(\zeta = \infty) = 1 \quad \text{for q.e. } x \in X.$$

This means that the particle stays at X forever. We also see that if $(\mathcal{E}, \mathcal{F})$ is transient, then

$$P_x\left(\zeta = \infty \text{ and } \lim_{t\to\infty} X_t = \Delta\right) = P_x(\zeta = \infty) \quad \text{for q.e. } x \in X \tag{2.4}$$

(e.g., see [5, Theorem 3.5.2]). This says that the particle escapes to infinity eventually under the survival event $\{\zeta = \infty\}$.

A set $B \subset X_\Delta$ is said to be *nearly Borel measurable* if for any $\mu \in \mathcal{P}(X_\Delta)$, there exist $B_1, B_2 \in \mathcal{B}(X_\Delta)$ such that $B_1 \subset B \subset B_2$ and

$$P_\mu(X_t \in B_2 \setminus B_1 \text{ for some } t > 0) = 0.$$

A set $N \subset X$ is said to be *properly exceptional* if $m(N) = 0$ and $X \setminus N$ is $\mathbf{M}$-*invariant*, that is,

$$P_x\left(X_t \in (X \setminus N)_\Delta \text{ and } X_{t-} \in (X \setminus N)_\Delta \text{ for any } t > 0\right) = 1, \quad x \in X \setminus N.$$

Here $(X \setminus N)_\Delta = (X \setminus N) \cup \{\Delta\}$ and $X_{t-} = \lim_{s \to t-0} X_s$. We note that $\mathrm{Cap}(N) = 0$ for any properly exceptional set N ([12, Theorem 4.2.1]).

§ 3. Upper rate functions

In this section, we discuss upper rate functions for symmetric Markov processes by following [39]. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$ and $\mathbf{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in X})$ an associated m-symmetric Hunt process on X. Throughout this paper, we impose the following assumption on $(\mathcal{E}, \mathcal{F})$.

Assumption 1.

(i) The killing measure k vanishes;

(ii) There exists an integral kernel $J(x, \mathrm{d}y)$ such that $J(\mathrm{d}x\mathrm{d}y) = J(x, \mathrm{d}y)\, m(\mathrm{d}x)$.

Under this assumption, we have

$$\mathcal{E}(u, u) = \frac{1}{2}\mu^c_{\langle u \rangle}(X) + \iint_{X \times X \setminus \mathrm{diag}} (u(x) - u(y))^2\, J(x, \mathrm{d}y) m(\mathrm{d}x).$$

In particular, $\mathbf{M}$ has no killing inside in the sense that

$$P_x\left(\zeta < \infty, X_{\zeta-} \in X\right) = 0 \quad \text{for q.e. } x \in X.$$

According to [45], we introduce a class $\mathcal{A}$ of functions which measure the positions of particles. Let

$$\mathcal{F}_{\mathrm{loc,\,ac}} = \left\{\rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X) \mid \mu^c_{\langle \rho \rangle} \text{ is absolutely continuous with respect to } m\right\}.$$

For $\rho \in \mathcal{F}_{\mathrm{loc,\,ac}}$, we denote by $\Gamma^c(\rho)$ the density function of $\mu^c_{\langle \rho \rangle}$ with respect to the measure m. We set $B_\rho(r) = \{x \in X \mid \rho(x) < r\}$ for $r > 0$ and

$$\mathcal{A} = \left\{\rho \in \mathcal{F}_{\mathrm{loc,\,ac}} \mid \lim_{x \to \Delta} \rho(x) = \infty \text{ and } B_\rho(r) \text{ is relatively compact for each } r > 0\right\}.$$

We assume that $\mathcal{A}$ is not empty and fix $\rho \in \mathcal{A}$. Then by the definition of $\mathcal{A}$,

$$\{\zeta = \infty\} = \{\rho(X_t) < \infty \text{ for any } t > 0\}.$$

Here we are interested in the upper bound of $\rho(X_t)$ for all sufficiently large $t > 0$. More precisely, we would like to find a positive increasing function $R(t)$ on $(0, \infty)$ such that

$$P_x\left(\text{there exists } T > 0 \text{ such that } \rho(X_t) \leq R(t) \text{ for all } t \geq T\right) = 1, \quad \text{q.e. } x \in X.$$

Such function is called an *upper rate function* for $\mathbf{M}$ with respect to ρ.

We now introduce a result on upper rate functions for symmetric Markov processes with no killing inside. Let $v(r)$ be a nondecreasing function on $(0,\infty)$ such that $m(B_\rho(r)) \le v(r)$ for any $r > 0$ and

$$C_R = \frac{1}{32} \cdot \frac{R}{\log v(R) + \log\log R} \quad \text{for } R \ge 6.$$

Assumption 2. Let $\rho_1 := \rho$. There exists a pair of a nondecreasing sequence of symmetric positive functions $F_r(x,y)$ $(r \ge 1)$ on $X \times X \setminus \mathrm{diag}$ and a nondecreasing sequence of functions $\rho_r \in \mathcal{A}$ $(r \ge 1)$ such that

(i) For each $r > 0$,

$$\sup_{x\in X} \int_{d(x,y)\ge F_r(x,y)} J(x,\mathrm{d}y) < \infty;$$

(ii) For each $r \ge 1$, $\sup_{0<d(x,y)<F_r(x,y)} |\rho_r(x) - \rho_r(y)|$ is finite. Moreover, there exists a constant $r_0 \ge 6$ such that for all $r \ge r_0$, if $0 < d(x,y) < F_r(x,y)$, then

$$|\rho_r(x) - \rho_r(y)| \le C_r;$$

(iii) For any compact set $K \subset X$, there exists a constant $r_1 = r_1(K) \ge 1$ such that $K \subset B_{\rho_r}(r/4)$ for any $r \ge r_1$.

Fix $r > 0$. When a particle at $x \in X$ jumps to $y \in X$, we regard this jump as a relatively small (resp. big) jump if $d(x,y) < F_r(x,y)$ (resp. $d(x,y) \ge F_r(x,y)$). Then the condition (i) means that the intensity of relatively big jumps is small enough. The condition (ii) implies that the jump range of $\{\rho_r(X_t)\}_{t\ge 0}$ is bounded by C_r for all relatively small jumps. We make the technical condition (iii) because $B_{\rho_r}(r/4)$ is not necessarily monotone with respect to r.

We fix a pair of sequences of functions $F_r(x,y)$ and ρ_r satisfying Assumption 2 and let

$$\Gamma_r^j(u)(x) = \int_{0<d(x,y)<F_r(x,y)} (u(x) - u(y))^2 \, J(x,\mathrm{d}y), \quad x \in X.$$

We define

$$M_1^r(u,R) = \operatorname*{ess\,sup}_{x\in B_{\rho_r}(R)} \Gamma^c(u)(x) + \operatorname*{ess\,sup}_{x\in B_{\rho_r}(R)} \Gamma_r^j(u)(x)$$

and $M_1(u,R) = M_1^R(u,R)$ for $R \ge 1$. Let $N_1(R)$ be a nondecreasing function on $[r_0,\infty)$ such that $N_1(R) \ge 1 \vee M_1(\rho_R, R)$ for any $R \ge r_0$, and put

$$N_2(r) = \sup_{x\in X} \int_{d(x,y)\ge F_r(x,y)} J(x,\mathrm{d}y).$$

For fixed $\varepsilon > 0$, we define the function $\psi_\varepsilon(R)$ $(R \ge 6)$ by

$$\psi_\varepsilon(R) = \frac{R^2}{N_1(R)(\log v(R) + \log\log R)} \wedge \frac{1}{N_2(R)(\log R)^{1+\varepsilon}}. \tag{3.1}$$

Assumption 3. There exists a constant $r_2 \geq r_0$ such that $\psi_\varepsilon(R)$ is strictly increasing on (r_2, ∞) and $\lim_{R\to\infty} \psi_\varepsilon(R) = \infty$.

Theorem 3.1. ([38]) If $(\mathcal{E}, \mathcal{F})$ is conservative and Assumptions 1–3 are fulfilled, then there exists a positive constant c such that $\psi_\varepsilon^{-1}(ct)$ is an upper rate function for $\mathbf{M}$ with respect to ρ.

For symmetric diffusion processes, similar results are obtained by replacing $\psi_\varepsilon(R)$ with

$$\psi(R) = \int_6^R \frac{r}{N_1(r)(\log v(r) + \log\log r)}\, \mathrm{d}r \tag{3.2}$$

for $\rho_R = \rho$. The result of this type was first proved by Grigor'yan [16] for the Brownian motion on a complete Riemannian manifold M. Here d is the Riemannian distance and $\rho(x) = d(o, x)$ for a fixed point $o \in M$. However, the function $\psi(R)$ there was given by

$$\psi(R) = \frac{R^2}{\log m(B_\rho(R))} \tag{3.3}$$

with the Riemannian volume m. Therefore, we could not allow the volume growth condition like $m(B_\rho(R)) \asymp e^{cR^2}$ $(c > 0)$. This restriction was relaxed by Grigor'yan-Hsu [17] under the Sobolev inequality. Hsu-Qin [21] obtained upper rate functions in terms of the function $\psi(R)$ in (3.2) without the Sobolev inequality; they applied the Lyons-Zheng decomposition to ρ by following the idea of Takeda [45] (see also [12, 5.7]). Ouyang [37] developed the approach of [21] to symmetric diffusion processes generated by regular Dirichlet forms. Similar results are further proved for Markov chains on weighted graphs ([22, 24]).

Theorem 3.1 is an extension of the results mentioned above to symmetric Hunt processes with no killing inside. Since the first term of $\psi_\varepsilon(R)$ in (3.1) is similar to $\psi(R)$ in (3.3), Theorem 3.1 is not sharp for symmetric diffusion processes and Markov chains on weighted graphs. The second term of $\psi_\varepsilon(R)$ in (3.1) comes from the intensity of big jumps.

Remark. Suppose that for a fixed point $x_0 \in X$, the function $d_0(x) = d(x, x_0)$ belongs to the class $\mathcal{A}$. If Assumptions 2–3 are satisfied with $\rho = d_0$, then by Theorem 3.1 and the triangle inequality, there exists $c > 0$ such that

$$P_x\left(\text{there exists } T > 0 \text{ such that } d(x, X_t) \leq \psi_\varepsilon^{-1}(ct) \text{ for all } t \geq T\right) = 1 \quad \text{for q.e. } x \in X. \tag{3.4}$$

Hence $\psi_\varepsilon^{-1}(ct)$ is an upper rate function in the sense of [16, 17, 21, 22, 24, 37] for q.e. starting point $x \in X$.

We now make a comment on the proof of Theorem 3.1. Our approach is similar to that of [16, 17, 22, 24]. We suppose that Assumptions 1–3 are fulfilled. Fix $\theta \in (1,2)$ and $R(t) = \psi_\varepsilon^{-1}(ct)$ for $c = 1024$. We define the event A_n by

$$A_n = \{\rho(X_t) \geq R(t) \text{ for some } t \in (t_n, t_{n+1}]\}$$

for $t_n = \theta^n$. If we can show that

$$\sum_{n=0}^{\infty} P_x(A_n) < \infty, \tag{3.5}$$

then the proof is complete by the Borel-Cantelli lemma.

Let $R_n = R(t_n)$. Then

$$P_x(A_n) \leq P_x\left(\sup_{0<t\leq t_{n+1}} \rho(X_t) \geq R_n\right) = P_x\left(\tau_{B_\rho(R_n)} \leq t_{n+1}\right).$$

Here $\tau_G = \inf\{t > 0 \mid X_t \notin G\}$ is the exit time of $\mathbf{M}$ from an open set $G \subset X$. We then want to estimate the probability $P_x(\tau_{B_\rho(R)} \leq t)$. If $\mathbf{M}$ is a symmetric diffusion process, then the particle sits on the boundary of $B_\rho(R)$ when leaving from $B_\rho(R)$. However, since we allow the Dirichlet form $(\mathcal{E}, \mathcal{F})$ to have the jumping measure in general, the particle can sit outside $\overline{B_\rho(R)}$ at time $\tau_{B_\rho(R)}$. Thus we can not directly give an estimate of the probability $P_x(\tau_{B_\rho(R)} \leq t)$ by following [16, 17, 22, 24]. In order to avoid this difficulty, we decompose $(\mathcal{E}, \mathcal{F})$ into the small and big jump parts. Namely, we define $\mathcal{E}^{(R,1)}$ and $\mathcal{E}^{(R,2)}$, respectively, by

$$\begin{aligned}
\mathcal{E}^{(R,1)}(u,u) &= \frac{1}{2}\mu^c_{\langle u\rangle}(X) + \iint_{0<d(x,y)<F_R(x,y)} (u(x)-u(y))^2 J(x,\mathrm{d}y)m(\mathrm{d}x),\\
\mathcal{E}^{(R,2)}(u,u) &= \iint_{d(x,y)\geq F_R(x,y)} (u(x)-u(y))^2 J(x,\mathrm{d}y)m(\mathrm{d}x).
\end{aligned}$$

A similar decomposition as above is used in [18, 32, 33, 38, 41] for the conservativeness criteria. Since the jumping measure $J(\mathrm{d}x\mathrm{d}y) = J(x,\mathrm{d}y)m(\mathrm{d}x)$ is symmetric, we obtain

$$\mathcal{E}_1^{(R,1)}(u,u) \asymp \mathcal{E}_1(u,u), \quad u \in \mathcal{F}$$

by Assumption 2. Hence $(\mathcal{E}^{(R,1)}, \mathcal{F})$ is also a regular Dirichlet form on $L^2(X;m)$ so that there exists an associated m-symmetric Hunt process $\mathbf{M}^{(R)} = \left(\{X_t^{(R)}\}_{t\geq 0}, \{P_x^{(R)}\}_{x\in X}\right)$. Furthermore, we can regard $\left\{\rho_R\left(X_t^{(R)}\right)\right\}_{t\geq 0}$ as a jump process with finite jump range less than C_R. Then in a similar way to [16, 17, 22, 24], we can get an L^2-estimate of the probability

$$u_R(t,x) = P_x^{(R)}\left(\tau^R_{B_{\rho_R}(R-C_R)} \leq t\right),$$

where $\tau_G^R = \inf\left\{t > 0 \mid X_t^{(R)} \notin G\right\}$ is the exit time of $\mathbf{M}^{(R)}$ from an open set $G \subset X$. Note that by Assumption 2, the particle is inside $B_{\rho_R}(R)$ when leaving from $B_{\rho_R}(R - C_R)$.

We can further construct (an equivalent version of) $\mathbf{M}$ by adding big jumps to $\mathbf{M}^{(R)}$ as an application of the so-called *Ikeda-Nagasawa-Watanabe's piecing out* ([28]) or *Meyer's construction* of Markov processes ([34]). Roughly speaking, the trajectories of sample paths of $\mathbf{M}$ and $\mathbf{M}^{(R)}$ are the same until the first big jump time of $\mathbf{M}$. We then obtain an inequality

$$P_x(\tau_{B_\rho(R)} \le t) \le P_x(\tau_{B_{\rho_R}(R-C_R)} \le t) \le u_R(t,x) + 2tN_2(R) \tag{3.6}$$

by noting that $\rho_R(x) \ge \rho(x)$ for any $x \in X$. The first term of the right hand side of (3.6) comes from the probability that the particle exits from $B_{\rho_R}(R - C_R)$ before time t and no big jump occurs until time t. The second term comes from the probability that the big jump occurs until time t. Using (3.6), we can show (3.5).

See, e.g., [1, 2, 3, 7] for applications of Ikeda-Nagasawa-Watanabe's piecing out and Meyer's construction of Markov processes to jump processes.

Example 3.2. ([38]) Let $B_x(r) = \{y \in X \mid d(y,x) < r\}$ be an open ball in X with center $x \in X$ and radius $r > 0$. Suppose that $B_x(r)$ is relatively compact for any $x \in X$ and $r > 0$, and there exist $\alpha > 0$ and $c_0 > 0$ such that

$$m(B_x(r)) \le c_0 r^\alpha$$

for any $x \in X$ and $r > 0$.

We denote by $C_0^{\mathrm{lip}}(X)$ the totality of Lipschitz continuous functions on X with compact support. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$ such that $C_0^{\mathrm{lip}}(X) \subset \mathcal{F}$ and

$$\mathcal{E}(u,v) = \iint_{X \times X \setminus \mathrm{diag}} (u(x) - u(y))(v(x) - v(y))\, J(x,y) m(\mathrm{d}y) m(\mathrm{d}x), \quad u, v \in \mathcal{F},$$

where $J(x,y)$ is a symmetric and positive function on $X \times X \setminus \mathrm{diag}$. Then for any fixed point $x_0 \in X$, the function $d_0(x) = d(x_0, x)$ belongs to $\mathcal{F}_{\mathrm{loc}}$.

Let $\mathbf{M} = (\{X_t\}_{t \ge 0}, \{P_x\}_{x \in X})$ be an m-symmetric Hunt process on X generated by $(\mathcal{E}, \mathcal{F})$. Then Theorem 3.1 implies the next assertions:

(i) We first assume that, for some $\beta_1 \in (0,2)$, $\beta_2 > 0$ and $\lambda > 0$,

$$J(x,y) \lesssim \frac{1}{d(x,y)^{\alpha+\beta_1}} \mathbf{1}_{\{0 < d(x,y) < 1\}} + \frac{e^{-\lambda d(x,y)}}{d(x,y)^{\beta_2}} \mathbf{1}_{\{d(x,y) \ge 1\}}.$$

Then there exists $c > 0$ such that (3.4) is valid by replacing $\psi_\varepsilon^{-1}(t)$ with $R(t) = c\sqrt{t \log t}$.

(ii) We next assume that for some $\beta_1 \in (0,2)$ and $\beta_2 \in (0,2)$,

$$J(x,y) \lesssim \frac{1}{d(x,y)^{\alpha+\beta_1}}\mathbf{1}_{\{0<d(x,y)<1\}} + \frac{1}{d(x,y)^{\alpha+\beta_2}}\mathbf{1}_{\{d(x,y)\geq 1\}}.$$

Then there exists $c>0$ such that (3.4) is valid by replacing $\psi_\varepsilon^{-1}(t)$ with $R(t) = ct^{\frac{1}{\beta_2}}(\log t)^{1+\frac{1+\varepsilon}{\beta_2}}$ for any $\varepsilon > 0$.

Remark.

(i) The upper rate function in Example 3.2 (i) is similar to that for symmetric diffusion processes associated with Dirichlet forms in Example 2.1 (see [37]).

(ii) We assume that, for some $\alpha > 0$ and $\beta \in (0,2)$,

$$J(x,y) \asymp \frac{1}{d(x,y)^{\alpha+\beta}}$$

for any $(x,y) \in X \times X \setminus \mathrm{diag}$ and

$$m(B_x(r)) \asymp r^\alpha$$

for any $x \in X$ and $r > 0$. Then an associated symmetric Hunt process is the so-called β-symmetric stable-like process in the sense of Chen-Kumagai [6]. It follows by [42] that

(3.7)

$$P_x\,(\text{there exists } T>0 \text{ such that } d(x,X_t) \leq R(t) \text{ for all } t \geq T) = \begin{cases} 1 & (\varepsilon > 0) \\ 0 & (\varepsilon \leq 0) \end{cases}$$

for $R(t) = t^{\frac{1}{\beta}}(\log t)^{\frac{1+\varepsilon}{\beta}}$. This result is first proved by Khintchine [30] for symmetric stable processes. We see from (3.7) that the upper rate function in Example 3.2 (ii) is not sharp in general. However, we do not know whether it is possible to obtain a similar result merely by the upper bounds of the rates of the volume growth and jumps.

§ 4. Lower rate functions

In this section, we discuss by following [40] the lower rate functions for symmetric Markov processes generated by regular Dirichlet forms. Let $(\mathcal{E},\mathcal{F})$ be a transient regular Dirichlet form on $L^2(X;m)$ and $\mathbf{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in X})$ an associated m-symmetric Hunt process on X. We then see from (2.4) that the particle escapes to infinity as time goes to infinity. If we further assume that $\mathcal{A}$ is nonempty and fix $\rho \in \mathcal{A}$, then

$$\left\{\zeta = \infty, \lim_{t\to\infty} X_t = \Delta\right\} = \left\{\zeta = \infty, \lim_{t\to\infty} \rho(X_t) = \infty\right\}.$$

Here we would like to find the lower bound of $\rho(X_t)$ for all sufficiently large $t > 0$. A positive increasing function $r(t)$ on $(0,\infty)$ is called a *lower rate function* for $\mathbf{M}$ with respect to ρ if

$$P_x(\text{there exists } T > 0 \text{ such that } \rho(X_t) \geq r(t) \text{ for all } t \geq T) = 1 \quad \text{for q.e. } x \in X.$$

Such lower bound expresses the speed of particles escaping to infinity.

Let us introduce a result on lower rate functions for $\mathbf{M}$. We make the next assumption on the transition function $p_t(x, \mathrm{d}y)$ of $\mathbf{M}$.

Assumption 4. *There exist a properly exceptional Borel set $N \subset X$ and a nonnegative symmetric kernel $p_t(x,y)$ on $(0,\infty) \times (X \setminus N) \times (X \setminus N)$ such that*

$$p_t(x, A) = \int_E p_t(x,y)\, m(\mathrm{d}y), \quad t \geq 0,\ E \in \mathcal{B}(X)$$

for any $x \in X \setminus N$ and

$$p_{t+s}(x,y) = \int_X p_t(x,z) p_s(z,y)\, m(\mathrm{d}z) \tag{4.1}$$

for any $x, y \in X \setminus N$ and $t > 0$.

Under Assumption 4, we define $p_t(x,y) = 0$ for $(x,y) \notin (X \setminus N) \times (X \setminus N)$. We know from [1, Theorem 3.1] that if there exists a left continuous positive function M on $(0,\infty)$ such that $\|T_t f\|_\infty \leq M(t)\|f\|_1$, then Assumption 4 is fulfilled and

$$p_t(x,y) \leq M(t)$$

for any $x, y \in X \setminus N$ and $t > 0$.

In what follows, we assume that the class $\mathcal{A}$ is not empty and fix $\rho \in \mathcal{A}$. Define

$$w^{(c)}(R) = \operatorname*{ess\,sup}_{x \in B_\rho(R)} \Gamma^c(\rho)(x)$$

and

$$w^{(j)}(R) = \operatorname*{ess\,sup}_{x \in X} \int_{X \setminus \{x\}} \left\{ (\rho(x) - \rho(y))^2 \wedge R^2 \right\} J(x, \mathrm{d}y).$$

Let $v(r)$ be a nondecreasing function on $(0,\infty)$ such that $m(B_\rho(r)) \leq v(r)$ for all $r > 0$ and $g(r)$ a differentiable and nonincreasing function on $(0,\infty)$ such that

$$\frac{1}{r^2}(w^{(c)}(r) + w^{(j)}(r)) \leq g(r)$$

for all $r > 0$. Define $h(r) = 1/g(r)$ and

$$I(R) = \int_R^\infty \frac{h'(t)}{v(t)}\, \mathrm{d}t.$$

This function is related to the recurrence of $(\mathcal{E},\mathcal{F})$ (see, e.g., [36] and references therein). For instance, let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(\mathbb{R}^d)$ generated by the symmetric stable process with index $\alpha\in(0,2)$:

$$\mathcal{F}=\left\{u\in L^2(\mathbb{R}^d)\mid \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y<\infty\right\},$$
$$\mathcal{E}(u,v)=c_{d,\alpha}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,\mathrm{d}x\mathrm{d}y$$

for

$$c_{d,\alpha}=\frac{\alpha 2^{\alpha-2}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}.$$

If we take $\rho(x)=|x|\in\mathcal{A}$, then $w^{(j)}(r)\le c_1r^{2-\alpha}$ for some $c_1>0$ so that we can take $h(r)=c_2r^{\alpha}$ with some $c_2>0$. Hence we have

$$I(R)=\begin{cases}\dfrac{c_3}{R^{d-\alpha}}, & d>\alpha\\ \infty, & d\le\alpha\end{cases}$$

for some $c_3>0$. Here we recall that $(\mathcal{E},\mathcal{F})$ (or the symmetric α-stable process on $\mathbb{R}^d$) is recurrent if and only if $d\le\alpha$ (See, e.g., [12, Example 1.5.2]).

We assume the volume doubling condition as follows:

Assumption 5. (Volume doubling condition) There exists $c_V>0$ such that for all $R>0$,

$$m(B_\rho(2R))\le c_V\cdot m(B_\rho(R)).$$

Theorem 4.1. ([40]) Let $(\mathcal{E},\mathcal{F})$ be a transient regular Dirichlet form on $L^2(X;m)$. Suppose that Assumptions 4–5 are fulfilled and $I(R)<\infty$ for all $R>0$. Then a positive increasing function $r(t)$ on $(0,\infty)$ is a lower rate function for $\mathbf{M}$ with respect to ρ if

$$\int_{t_0}^{\infty}\frac{1}{I(r(s))}\sup_{y\in X}p_s(x,y)\,\mathrm{d}s<\infty,\quad x\in X \tag{4.2}$$

for some $t_0>0$.

Grigor'yan [16] (see also [4]) obtained an integral test on lower rate functions for the Brownian motion on a complete Riemannian manifold. Theorem 4.1 is an extension of this result to symmetric Hunt processes with no killing inside. Furthermore, as will be mentioned below, Theorem 4.1 is sharp for symmetric stable(-like) processes.

We now sketch the proof of Theorem 4.1, which is similar to that of Grigor'yan [16]. Let $r(t)$ be a positive increasing function on $(0,\infty)$ and $\{t_n\}_{n=0}^{\infty}$ an increasing sequence such that $\lim_{n\to\infty}t_n=\infty$ and

$$A_n=\{\rho(X_t)\le r(t)\text{ for some }t\in(t_{n-1},t_n]\}.$$

If we can show that

$$\sum_{n=1}^{\infty} P_x(A_n) < \infty \tag{4.3}$$

for some $\{t_n\}_{n=0}^{\infty}$, then the proof is complete by the Borel-Cantelli lemma.

Since the function $r(t)$ is increasing in t, we have

$$\begin{aligned} A_n &\subset \{\rho(X_t) \le r(t_n) \text{ for some } t \in (t_{n-1}, t_n]\} \\ &\subset \{\rho(X_t) \le r(t_n) \text{ for some } t > t_{n-1}\} \\ &= \left\{X_t \in \overline{B_\rho(r(t_n))} \text{ for some } t > t_{n-1}\right\} \end{aligned} \tag{4.4}$$

so that

$$P_x(A_n) \le P_x\left(X_t \in \overline{B_\rho(r(t_n))} \text{ for some } t > t_{n-1}\right). \tag{4.5}$$

Here we estimate the right hand side of (4.5) by following the argument of Bendikov and Saloff-Coste [4, Theorem 3.10]. For a compact set K in X, we define

$$\psi_K(t,x) = P_x\left(X_s \in K \text{ for some } s > t\right), \quad x \in X \setminus N, \ t \ge 0.$$

Then by the Markov property and Assumption 4, we have

$$\begin{aligned} \psi_K(t,x) &= E_x\left[P_{X_t}\left(X_s \in K \text{ for some } s > 0\right)\right] \\ &= \int_X p_t(x,y) P_y\left(X_s \in K \text{ for some } s > 0\right) m(\mathrm{d}y). \end{aligned} \tag{4.6}$$

Let $G(x,y)$ be the Green function of $\mathbf{M}$ defined by

$$G(x,y) = \int_0^\infty p_t(x,y)\,\mathrm{d}t. \tag{4.7}$$

Then by combining [12, Theorem 4.3.3] with [5, Lemma 6.1.1], we obtain

$$P_y(X_t \in K \text{ for some } t > 0) = \int_X G(y,z)\nu_K(\mathrm{d}z) \quad \text{for } m\text{-a.e. } y \in X, \tag{4.8}$$

where ν_K is the equilibrium measure of K. Moreover, since $\nu_K(K) = \mathrm{Cap}_{(0)}(K)$ and

$$\int_X p_t(x,y)G(y,z)\,m(\mathrm{d}y) = \int_t^\infty p_s(x,z)\,\mathrm{d}s$$

by (4.1), it follows from (4.7) and (4.8) that the last term of (4.6) is less than

$$\mathrm{Cap}_{(0)}(K) \int_t^\infty \sup_{y \in X} p_s(x,y)\,\mathrm{d}s, \quad t > 0, \ x \in X \setminus N.$$

We can further obtain the capacitary upper bound as an application of the result by Ôkura [35]: if $I(R) < \infty$ for any $R > 0$, then there exists $C > 0$ such that for any $R > 0$,

$$\mathrm{Cap}(\overline{B_\rho(R)}) \le \frac{C}{I(R)}.$$

As a consequence of the argument above, we obtain

$$\begin{aligned} P_x(A_n) &\le \psi_{\overline{B_\rho(r(t_n))}}(t_{n-1}, x) \le \mathrm{Cap}_{(0)}(\overline{B_\rho(r(t_n))}) \int_{t_{n-1}}^\infty \sup_{y\in X} p_s(x,y)\,\mathrm{d}s \\ &\le \frac{C}{I(r(t_n))} \int_{t_{n-1}}^\infty \sup_{y\in X} p_s(x,y)\,\mathrm{d}s. \end{aligned}$$

Furthermore, by taking a suitable sequence $\{t_n\}$, we see from (4.2) that

$$\begin{aligned} \sum_{n=1}^\infty P_x(A_n) &\le C \sum_{n=1}^\infty \frac{1}{I(r(t_n))} \int_{t_{n-1}}^\infty \sup_{y\in X} p_s(x,y)\,\mathrm{d}s \\ &\le C' \int_{t_0}^\infty \frac{1}{I(r(s))} \sup_{y\in X} p_s(x,y)\,\mathrm{d}s < \infty. \end{aligned}$$

Therefore, we arrive at (4.3) and finish the proof of Theorem 4.1.

Under some restricted condition, we can rewrite (4.2) as an integral similar to that on lower rate functions for the Brownian motion and symmetric stable processes.

Assumption 6. In addition to Assumptions 4–5, the next conditions hold:

(i) There exist $p > 0$ and $c_1 > 0$ such that

$$p_t(x,x) \le \frac{c_1}{v(t^p)}$$

for any $x \in X$ and $t \ge 1$;

(ii) There exist $\nu > 0$ and $c_2 > 0$ such that

$$\frac{v(R)}{v(r)} \ge c_2 \left(\frac{R}{r}\right)^\nu$$

for any $r > 0$ and $R > r$;

(iii) There exists $c_3 > 1$ such that

$$h(c_3 R) \ge 2h(R)$$

for any $R > 0$.

Corollary 4.2. ([40]) Let $(\mathcal{E},\mathcal{F})$ be a transient regular Dirichlet form on $L^2(X;m)$ satisfying Assumption 1. Suppose that Assumption 6 is fulfilled and $I(R)<\infty$ for all $R>0$. Let $r(t)$ be a positive strictly increasing function on $(0,\infty)$ such that $r(t)/t^p \to 0$ $(t\to\infty)$ and

$$\int_{t_0}^{\infty} \frac{r(t)^{\nu}}{t^{p\nu}h(r(t))}\,\mathrm{d}t<\infty$$

for some $t_0>0$. Then $r(t)$ is a lower rate function for $\mathbf{M}$ with respect to ρ.

Example 4.3. ([40]) Let $\alpha>0$. Suppose that for any $x\in X$ and $r>0$, $B_x(r)=\{y\in X \mid d(y,x)<r\}$ is a relatively compact open set in X and $m(B_x(r))\asymp r^{\alpha}$. Let γ be a Borel measurable function on $X\times X$ such that

$$\begin{aligned}\beta_1\le\gamma(x,y)\le\beta_2,\quad d(x,y)<1,\\ \gamma_1\le\gamma(x,y)\le\gamma_2,\quad d(x,y)\ge 1\end{aligned}$$

for some constants $\beta_1,\beta_2,\gamma_1,\gamma_2\in(0,2)$ with $\beta_1\le\beta_2$ and $\gamma_1\le\gamma_2$. Let $J(x,y)$ be a positive symmetric function on $X\times X\setminus\mathrm{diag}$ such that

$$J(x,y)\asymp\frac{1}{d(x,y)^{\alpha+\gamma(x,y)}}$$

and $(\mathcal{E},\mathcal{F})$ a (regular) Dirichlet form on $L^2(X;m)$ such that $C_0^{\mathrm{lip}}(X)\subset\mathcal{F}$ and

$$\mathcal{E}(u,u)=\iint_{X\times X\setminus\mathrm{diag}}(u(x)-u(y))^2\,J(x,y)m(\mathrm{d}x)m(\mathrm{d}y),\quad u\in\mathcal{F}\cap C_0(X).$$

Then for any fixed point $o\in X$, the function $\rho(x)=d(o,x)$ belongs to $\mathcal{F}_{\mathrm{loc}}\cap C(X)$. Since

$$\begin{aligned}&\mathcal{E}(u,u)\\ &\ge c\left(\iint_{d(x,y)<1}\frac{(u(x)-u(y))^2}{d(x,y)^{\alpha+\beta_1}}\,m(\mathrm{d}x)m(\mathrm{d}y)+\iint_{d(x,y)\ge 1}\frac{(u(x)-u(y))^2}{d(x,y)^{\alpha+\gamma_2}}\,m(\mathrm{d}x)m(\mathrm{d}y)\right)\end{aligned}$$

for some $c>0$, we see from [1, Theorem 3.1] and [7, Theorems 3.1 and 3.2] that there exists a properly exceptional Borel set $N\subset X$ such that

$$p_t(x,\mathrm{d}y)=p_t(x,y)m(\mathrm{d}y)$$

for some positive symmetric kernel $p_t(x,y)$ on $(0,\infty)\times(X\setminus N)\times(X\setminus N)$ satisfying

$$p_t(x,y)\le\frac{c}{t^{\alpha/\gamma_2}}\quad\text{for all } t\ge 1.$$

We assume that $0<\gamma_1\le\gamma_2<2\wedge\alpha$. Then $(\mathcal{E},\mathcal{F})$ is transient because

$$\int_1^{\infty}p_t(x,y)\,\mathrm{d}t<\infty$$

(see, e.g., [40, Remark 2.5]). Fix $\rho(x) = d(o,x)$ for some $o \in X$. We can then take $h(t) = ct^{\gamma_1}$ so that

$$\int_e^\infty \frac{r(t)^\alpha}{t^{\alpha/\gamma_2} h(r(t))}\,\mathrm{d}t \asymp \int_e^\infty \frac{r(t)^\alpha}{t^{\alpha/\gamma_2} r(t)^{\gamma_1}}\,\mathrm{d}t = \int_e^\infty \frac{r(t)^{\alpha-\gamma_1}}{t^{\alpha/\gamma_2}}\,\mathrm{d}t. \tag{4.9}$$

Hence by using Corollary 4.2 with $\nu = \alpha$ and $p = 1/\gamma_2$, we see that for any $\varepsilon > 0$ and $c > 0$, the function

$$r(t) = \frac{ct^{\frac{1}{\gamma_2}\cdot\frac{\alpha-\gamma_2}{\alpha-\gamma_1}}}{(\log t)^{\frac{1+\varepsilon}{\alpha-\gamma_1}}}$$

is a lower rate function for $\mathbf{M}$ with respect to ρ.

When $\gamma_1 = \gamma_2 (= \gamma)$, the right hand side of (4.9) appears in the 0-1 law-type integral test on lower rate functions for symmetric γ-stable(-like) processes (see [47] and [42]). Therefore, Theorem 4.1 and Corollary 4.2 are sharp for symmetric stable(-like) processes. Here we should mention that (4.9) is independent of β_1 and β_2. Hence the lower rate function is not affected by replacing the index β_1 with the smaller one.

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